3-D forward modeling of elastic wave equation:

Stress-Displacement formulation, see e.g. Graves et al. (1996), equations (1-2) which include:

Equation of momentum conservation

$$
\begin{aligned}
\rho \ddot{u_x} &= \partial_x \tau_{xx} + \partial_y \tau_{xy} + \partial_z \tau_{xz} + f_x \\
\rho \ddot{u_y} &= \partial_x \tau_{xy} + \partial_y \tau_{yy} + \partial_z \tau_{yz} + f_x \\
\rho \ddot{u_z} &= \partial_x \tau_{xz} + \partial_y \tau_{yz} + \partial_z \tau_{zz} + f_z
\end{aligned}
$$

And Equation of stress-strain relations

$$
\begin{aligned}
\tau_{xx} &= (\lambda + 2\mu)\partial_x u_x + \lambda(\partial_y u_y + \partial_z u_z) \\
\tau_{yy} &= (\lambda + 2\mu)\partial_y u_y + \lambda(\partial_x u_x + \partial_z u_z) \\
\tau_{zz} &= (\lambda + 2\mu)\partial_z u_z + \lambda(\partial_x u_x + \partial_y u_y) \\
\tau_{xy} &= \lambda(\partial_y u_x + \partial_x u_y) \\
\tau_{xz} &= \lambda(\partial_z u_x + \partial_x u_z) \\
\tau_{yz} &= \lambda(\partial_z u_y + \partial_y u_z)
\end{aligned}
$$

where (u_x, u_y, u_z) are the displacement components; $(\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz})$ are the stress components;(f_x, f_y, f_z) are the body-force components; ρ is the density; λ and μ are Lame coefficients. The symbols $\partial_x, \partial_y, \partial_z$ are the spatial differential operators and $\ddot{u}$ is the second order temporal derivative.

Stress-Velocity formulation, see e.g. Graves et al. (1996), equations (3-4) which include:

$$
\begin{aligned}
\dot{v_x} &= b(\partial_x \tau_{xx} + \partial_y \tau_{xy} + \partial_z \tau_{xz} + f_x) \\
\dot{v_y} &= b(\partial_x \tau_{xy} + \partial_y \tau_{yy} + \partial_z \tau_{yz} + f_y) \\
\dot{v_z} &= b(\partial_x \tau_{xz} + \partial_y \tau_{yz} + \partial_z \tau_{zz} + f_z)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\tau}_{xx} &= (\lambda + 2\mu)\partial_x v_x + \lambda(\partial_y v_y + \partial_z v_z) \\
\dot{\tau}_{yy} &= (\lambda + 2\mu)\partial_y v_y + \lambda(\partial_x v_x + \partial_z v_z) \\
\dot{\tau}_{zz} &= (\lambda + 2\mu)\partial_z v_z + \lambda(\partial_x v_x + \partial_y v_y) \\
\dot{\tau}_{xy} &= \lambda(\partial_y v_x + \partial_x v_y) \\
\dot{\tau}_{xz} &= \lambda(\partial_z v_x + \partial_x v_z) \\
\dot{\tau}_{yz} &= \lambda(\partial_z v_y + \partial_y v_z)
\end{aligned}
$$

where $b = 1/\rho$ is the buoyancy.

For convenience to storage **the strains for inversion**, the Stress-Displacement formulation also can be written in more general form as:

$$
\begin{aligned}
\rho \ddot{u}_i &= \partial_j \tau_{ij} + f_i \\
\tau_{ij} &= \lambda \theta \delta_{ij} + 2\mu \epsilon_{ij} \\
\epsilon_{ij} &= \frac{1}{2}(\partial_j u_i + \partial_i u_j)
\end{aligned}
$$

and the Stress-Velocity formulation can be written as:

$$
\begin{aligned}
\rho \dot{v}_i &= \partial_j \tau_{ij} + f_i \\
\dot{\tau}_{ij} &= \lambda \dot{\theta} \delta_{ij} + 2\mu \dot{\epsilon}_{ij} \\
\dot{\epsilon}_{ij} &= \frac{1}{2}(\partial_j v_i + \partial_i v_j)
\end{aligned}
$$

where i, j are spatial directions x,y and z; ϵ_{ij} are strain components; $\theta = \frac{\epsilon_{ii}}{3}$ is the mean strain; δ_{ij} is the Kronecker function. We also want to introduce **the deviator strains components** as following:

$$
\epsilon'_{ij} = \epsilon_{ij} - \theta.
$$

Finite-Difference Implementation

The equations of the Stress-Velocity formulation is solved using a staggered-grid finite-difference technique, see e.g. Graves et al. (1996), equations (5-6) which include:

Discrete form for the velocities

$$vx^{n+1/2}_{i+1/2,j,k} = vx^{n-1/2}_{i+1/2,j,k} + [\triangle tb(D_xTxx + D_yTxy + D_zTxz + f_x)]|^n_{i+1/2,j,k}$$
$$vy^{n+1/2}_{i,j+1/2,k} = vy^{n-1/2}_{i,j+1/2,k} + [\triangle tb(D_xTxx + D_yTxy + D_zTxz + f_x)]|^n_{i,j+1/2,k}$$
$$vz^{n+1/2}_{i,j,k+1/2} = vx^{n-1/2}_{i,j,k+1/2} + [\triangle tb(D_xTxx + D_yTxy + D_zTxz + f_x)]|^n_{i,j,k+1/2}$$

Discrete form for the stresses

$$Txx^{n+1}_{i,j,k} = Txx^n_{i,j,k} + \triangle t[(\lambda + 2\mu)(D_xvx + \lambda(D_xvy + D_zvz)]|^{n+1/2}_{i,j,k}$$
$$Tyy^{n+1}_{i,j,k} = Txx^n_{i,j,k} + \triangle t[(\lambda + 2\mu)(D_xvx + \lambda(D_xvy + D_zvz)]|^{n+1/2}_{i,j,k}$$
$$Txx^{n+1}_{i,j,k} = Txx^n_{i,j,k} + \triangle t[(\lambda + 2\mu)(D_xvx + \lambda(D_xvy + D_zvz)]|^{n+1/2}_{i,j,k}$$
$$Txy^{n+1}_{i+1/2,j+1/2,k} = Txy^n_{i+1/2,j+1/2,k} + \triangle t[\mu(D_xvy + D_yvx)]|^{n+1/2}_{i+1/2,j+1/2,k}$$
$$Txz^{n+1}_{i+1/2,j,k+1/2} = Txz^n_{i+1/2,j,k+1/2} + \triangle t[\mu(D_xvz + D_zvx)]|^{n+1/2}_{i+1/2,j,k+1/2}$$
$$Tyz^{n+1}_{i,j+1/2,k+1/2} = Tyz^n_{i,j+1/2,k+1/2} + \triangle t[\mu(D_yvz + D_zvy)]|^{n+1/2}_{i,j+1/2,k+1/2}$$

where the subscripts refer to the spatial indices, and the superscripts refer to the time index; the symbols D_x,D_y and D_z represent the discrete forms of the spatial differential operators ∂_x, ∂_y, and ∂_z

Storage of the wavefields

For wavefield storage, we suggest 3 options:

-Store the velocity wavefield icluding 3 components: vx, vy, vz (Time integration and spatial derivetives are required at later stages).

-Store the deformation fields icluding 9 components: D_xvx,D_yvy,D_zvz, D_xvy, D_yvx, D_xvz, D_zvx, D_yvz and D_zvy (Time integration is required at later stages).

-Store the deviator strain fields icluding 6 components: ϵ_{xx},ϵ_{yy},ϵ_{xy}, ϵ_{xz}, ϵ_{yz}, and θ.

For time subsampling, we can store the data fields at every n-th time step (n=5, 10, 20...)

For spatial domain downscale, (not mentioned yet!)