Adjoint source using cross-correlation between observed and synthetic seismograms

The cross-correlation traveltime shift and relative amplitude anomaly can be generalized to frequency-dependent quantities that completely describe the discrepancies between the target and synthetic waveforms.

Chen et. al 2007 presume the seismogram observed on the ith component of the rth receiver from the sth seismic source can be approximated by an instrument-filtered wavefield $u_i^s(x_r, t)$ calculated from an unknown (target) earth model m. The quality of the approximation will depend on the ambient seismic noise as well as inadequacies in the way m represents the actual Earth (signal-generated noise). For each seismogram, we consider a finite set of data functionals, indexed by n, that measure the misfit between $u_i^s(x_r, t)$ and the wavefield $\tilde{u}_i^s(x_r, t)$ synthesized from the starting model $\tilde{m}$:

$$d_{in}^{sr} = D_n[u_i^s(x_r, t), \tilde{u}_i^s(x_r, t)].$$

The target wave group on the observed instrument-filtered displacement is denoted as $f(t)$. The isolation filter $\tilde{f}(t)$, is synthesized to model the target displacement as well as any instrument filtering. The autocorrelation of **the isolation filter** is an even function of time and the cross-correlation between the isolation filter and the target wave group $f(t)$:

$$\tilde{C}_{ff}(t) = \tilde{f}(t) \otimes \tilde{f}(t).$$

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Following Gee Jordan (1992), we first apply a **Gaussian time window** of the form:

$$W(t) = \exp[-\frac{\sigma_\omega^2}{2}(t - t_c)^2].$$

onto both correlagrams to reduce the contributions of interfering wave groups to the observations.

For the symmetric autocorrelation, we set t_c. In general the cross-correlation is not symmetric and to minimize the signal distortion by windowing we usually centre the time window at the peak of the cross-correlagram. The windowed correlagrams are denoted as $W\tilde{C}_{ff}$ and WC_{ff}.

The next operation is the narrowband filtering which localizes the windowed correlagrams in the frequency domain. We consider **Gaussian narrow band filter** of the form:

$$F_i(\omega) = \exp[-\frac{(\omega-\omega_i)^2}{2\sigma_i^2}] + \exp[-\frac{(\omega+\omega_i)^2}{2\sigma_i^2}].$$

where the index i specifies a filter F_i with half-bandwidth σ_i and centre frequency ω_i.

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We obtain the approcimation of the windowed and narrowband filtered correlagrams in time domain:

$$FW\tilde{C}_{ff} \approx \tilde{g}(t) = \tilde{A}\exp[-\frac{\tilde{\sigma}_s^2(t-\tilde{t}_g)}{2}]\cos[\tilde{\omega}_s(t-\tilde{t}_p)].$$

$$FWC_{ff} \approx g(t) = A\exp[-\frac{\sigma_s^2(t-t_g)}{2}]\cos[\omega_s(t-t_p)].$$

And the GSDF measurements of phase-delay time and amplitude reduction time associated with filter F_i are determined as (Gee Jordan 1992)

$$\delta t_p = t_p - \tilde{t}_p$$

$$\delta t_q = -\frac{ln(A/\tilde{A})}{\tilde{\omega_s}}.$$

In frequency domain, the autocorrelation of the isolation filter can be expressed as:

$$\tilde{C}_{ff}(\omega) = \tilde{f}^*(\omega)\tilde{f}(\omega).$$

and the observed waveform can be expressed as a perturbation from the isolation filter as

$$f(\omega) = \tilde{f}(\omega) + \delta f(\omega).$$

where $\delta f(\omega)$ is the instrument-filtered displacement perturbation, which means the frequency-domain displacement m perturbation $\delta u_i^s(x_r, \omega)$, multiplied with the instrument response.

Thus the cross-correlation $C_{ff}(\omega)$ can be expressed as:

$$C_{ff}(\omega)(\omega) = \tilde{f}^*(\omega)f(\omega) = \tilde{C}_{ff}(\omega) + \tilde{f}^*(\omega)\delta f(\omega).$$

In the frequency domain, the synthetic can be mapped into the target by an **exponential (Rytov) operator**

$$f(\omega) = \tilde{f}^*(\omega)\exp i\omega[\delta\tau_p(\omega) + i\delta\tau_q(\omega)].$$

where the generalized seismological data functionals (GSDFs) are observational approximations to the phase-delay times $\delta\tau_p(\omega)$ and amplitude-reduction times $\delta\tau_q(\omega)$, obtained by time-windowing and narrow-band filtering of the waveform cross-correlagram (Gee Jordan 1992).

The exact Frchet derivative of $\delta_{p,q}$ with respect to the waveform $f(\omega)$ can therefore be obtained as:

$$\delta\tau_p(\omega) = \frac{1}{\omega}\Im[\frac{\tilde{f}^*(\omega)}{C_{ff}(\omega)(\omega)}\delta f(\omega)].$$

$$\delta\tau_q(\omega) = \frac{1}{\omega}\Re[\frac{\tilde{f}^*(\omega)}{C_{ff}(\omega)(\omega)}\delta f(\omega)].$$

Bring into the continuous forms, transform δf into the time domain and define a function $I(t)$ as

$$\text{I(t)}= \int_{-\infty}^{\infty} d\omega \exp(i\omega t)\left\{ H(\omega)\frac{\sigma_i'}{\sigma_w\tilde{A}}\exp[\frac{-(\omega-\omega_i)^2}{2(\sigma_w^2+\sigma_i^2)}]\frac{\tilde{f}*(\omega)}{\omega'}\right\}$$
$$+ \int_{-\infty}^{\infty} d\omega \exp(i\omega t)\left\{ H(\omega)\frac{\sigma_i'}{\sigma_w\tilde{A}}\exp[\frac{-(\omega+\omega_i)^2}{2(\sigma_w^2+\sigma_i^2)}]\frac{\tilde{f}*(\omega)}{\omega''}\right\}.$$

where $H(\omega)$ is **the Heaviside function** and $\omega' = \frac{\sigma_i^2\omega+\sigma_w^2\omega_i}{\sigma_i^2+\sigma_w^2}$, $\omega'' = \frac{\sigma_i^2\omega-\sigma_w^2\omega_i}{\sigma_i^2+\sigma_w^2}$

Finally, the exact Fre'chet derivatives of the observables $\delta t_{p,q}$ with respect to the time-domain waveform $f(t)$ can be expressed as

$$\delta tp, q = \int dt J_x(t) \delta f(t),$$

where **the seismogram perturbation kernel** $J_x(t)$ is given by $J_p(t) = \Im[I(t)]$, $J_q(t) = \Re[I(t)]$.